Diminishing inverse transfer and non-cascading dynamics in surface quasi-geostrophic turbulence

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Abstract

The inverse transfer in two-dimensional turbulence governed by the surface quasi-geostrophic (SQG) equation is studied. The nonlinear transfer of this system conserves the two quadratic quantities $\Psi_1 = \langle|(-\Delta)^{1/4}\psi|^2\rangle/2$ and $\Psi_2 = \langle|(-\Delta)^{1/2}\psi|^2\rangle/2$ (kinetic energy), where ψ is the streamfunction and $\langle\cdot\rangle$ denotes a spatial average. In the limit of infinite domain, the kinetic energy density Ψ_2 remains bounded. For power-law inverse-transfer region, the inverse flux of Ψ_1 diminishes as it proceeds toward sufficiently low wavenumbers, implying that no persistent inverse cascade of Ψ_1 is sustainable. The unrealizability of an inverse cascade of Ψ_1 implies that there is no direct cascade of Ψ_2. Hence, the dual-cascade picture which is widely believed to be realizable in two-dimensional Navier–Stokes turbulence does not apply to SQG turbulence. Numerical results supporting the theoretical predictions are presented.

Key words:
Surface quasi-geostrophic turbulence, Inverse transfer, Diminishing inverse flux
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1 Introduction

The motion of a three-dimensional stratified rotating fluid is characterized by the geostrophic balance between the Coriolis force and pressure gradient. The dynamics governed by the first order departure from this linear balance is known as quasi-geostrophic dynamics (see for example [4,5,20,22]), which can be described in terms of the (three-dimensional) geostrophic streamfunction $\psi(\mathbf{x}, t)$. The vertical dimension z is usually taken to be semi-infinite and

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doubly periodic conditions are usually imposed on the horizontal flow. Normally decay conditions are required as $z \to \infty$. At the flat surface boundary $z = 0$, the vertical gradient of $\psi(\mathbf{x}, t)$ matches the temperature field $T(\mathbf{x}, t)$, i.e. $T(\mathbf{x}, t)|_{z=0} = \partial_z \psi(\mathbf{x}, t)|_{z=0}$. For flows with zero potential vorticity, this surface temperature field can be identified with $(-\Delta)^{1/2}\psi$, where Δ is the (horizontal) two-dimensional Laplacian. The conservation equation governing the advection of the active temperature field $(-\Delta)^{1/2}\psi$ by the surface flow $\mathbf{u} = (-\partial_y \psi, \partial_x \psi)$ is [2,13,20,21]

$$\partial_t(-\Delta)^{1/2}\psi + J(\psi, (-\Delta)^{1/2}\psi) = 0, \tag{1}$$

where $J(\vartheta, \theta) = \partial_x \vartheta \, \partial_y \theta - \partial_x \theta \, \partial_y \vartheta$. Eq. (1) is known as the SQG equation.

In this paper, a forced-dissipative version of (1) is studied. A dissipative term of the form $\mu\Delta\psi$, where $\mu > 0$, which results from Ekman pumping at the surface, is considered (cf. [6,24]). Since $(-\Delta)^{1/2}\psi$ is the advected quantity, this physical dissipation mechanism corresponds to the hypoviscous dissipation operator $\mu(-\Delta)^{1/2}$. The dissipation coefficient μ has the dimensions of velocity and is not vanishingly small in the atmospheric context. The system is assumed to be driven by a forcing f. Thus, the forced-dissipative SQG equation can be written as

$$\partial_t(-\Delta)^{1/2}\psi + J(\psi, (-\Delta)^{1/2}\psi) = \mu\Delta\psi + f. \tag{2}$$

The Jacobian operator $J(\cdot, \cdot)$ admits the identities

$$\langle \phi J(\vartheta, \theta) \rangle = -\langle \vartheta J(\phi, \theta) \rangle = -\langle \theta J(\vartheta, \phi) \rangle, \tag{3}$$

where $\langle \cdot \rangle$ denotes the spatial average. As a consequence, the nonlinear term in (2) obeys the conservation laws

$$\langle \psi J(\psi, (-\Delta)^{1/2}\psi) \rangle = \langle (-\Delta)^{1/2}\psi J(\psi, (-\Delta)^{1/2}\psi) \rangle = 0. \tag{4}$$

It follows that the two quadratic quantities $\Psi_1 = \langle |(-\Delta)^{1/4}\psi|^2 \rangle/2$ and $\Psi_2 = \langle |(-\Delta)^{1/2}\psi|^2 \rangle/2$ (kinetic energy) are conserved by nonlinear transfer.

The simultaneous conservation of Ψ_1 and Ψ_2 by advective nonlinearities imposes strict constraints on the transfer (redistribution) of these quantities in wavenumber space. Let us consider the transfer of an amount $\Psi_1 = \epsilon$, initially distributed around a given wavenumber s, which corresponds to an initial $\Psi_2 = s\epsilon$. Let $\Psi_1(k)$ and $\Psi_2(k)$ be the resulting redistributions of $\Psi_1 = \epsilon$ and of $\Psi_2 = s\epsilon$. Given arbitrary wavenumbers $p < s$ and $q > s$, one has

$$\frac{1}{\epsilon}\int_q^\infty \Psi_1(k)\,\mathrm{d}k \le \frac{1}{q\epsilon}\int_q^\infty \Psi_2(k)\,\mathrm{d}k \le \frac{s}{q}, \tag{5}$$

$$\frac{1}{s\epsilon}\int_0^p \Psi_2(k)\,\mathrm{d}k \le \frac{p}{s\epsilon}\int_0^p \Psi_1(k)\,\mathrm{d}k \le \frac{p}{s}, \tag{6}$$

where the two conservation laws and straightforward inequalities have been used. The left-hand side of (5) [(6)] is the fraction of ϵ [$s\epsilon$] that gets transferred to wavenumbers $k \ge q$ [$k \le p$]. This fraction is bounded from above by s/q [p/s], implying that no significant fraction of ϵ [$s\epsilon$] can be transferred to wavenumbers $k \gg s$ [$k \ll s$]. This type of constraint on the nonlinear transfer of the invariants is a common feature in incompressible fluid systems in two dimensions. (Some familiar systems in this category are the Charney–Hasegawa–Mima equation [11,12] and the class of α turbulence equations [21], which includes both the Navier–Stokes and the SQG equations.) The implication is that when the said initial sources spread out in wavenumber space, Ψ_1 [Ψ_2] is preferentially transferred toward lower [higher] wavenumbers. According to the classical theory of two-dimensional turbulence [1,14,15,17], which was originally formulated for high-Reynolds number Navier–Stokes fluids and subsequently thought to apply to two-dimensional incompressible fluids in general, this preferential transfer achieves the extreme limit by transferring virtually all ϵ to $k \ll s$ (inverse cascade) and virtually all $s\epsilon$ to $k \gg s$ (direct cascade). The transfer of the invariants in this manner is known as the dual cascade. [1]

However, it is shown in this work that the preferential transfer of Ψ_1 and of Ψ_2 is not as dramatic as predicted but rather has a limited extent. More accurately, it is shown that an upper bound on the inverse flux of Ψ_1 across a low wavenumber ℓ vanishes (uniformly in time) as $\ell/s \to 0$, where s is the characteristic forcing wavenumber, thereby ruling out the existence of a persistent inverse cascade of Ψ_1 toward the low wavenumbers. The unrealizability of an inverse cascade implies that there is no direct cascade of Ψ_2 [24]. The physical reasons behind this behaviour are that the kinetic energy density of SQG dynamics remains bounded in the limit of infinite domain, a consequence of the hypoviscous dissipation operator $\mu(-\Delta)^{1/2}$, and that the inverse flux of Ψ_1 across ℓ is proportional to the energy content of the wavenumber region $k \le \ell$. These two properties of SQG turbulence, when combined, imply that the inverse flux of Ψ_1 becomes smaller for progressively lower ℓ, thus ruling out the existence of a persistent inverse cascade.

In the next section some preliminary estimates, which are employed in the

[1] For some recent discussion on the possibility of a dual cascade in various two-dimensional systems, including the Navier–Stokes and SQG equations, see [24,25,27,29].

derivations of the main results in Section 3, are presented. Section 4 examines the implications of the results in Section 3 for the long-time dynamics and spectral distribution of kinetic energy. Section 5 presents some numerical results in support of the theoretical prediction of no cascades in SQG dynamics. The paper ends with some discussion in the final section.

2 Preliminary estimates

This section presents a simple analytic inequality and reviews the boundedness of the energy density [24]. The former applies to bounded domains only, while the latter is valid for both bounded and unbounded cases.

For a doubly periodic domain $L \times L$, the (complex) Fourier representation of ψ is

$$\psi(\mathbf{x}) = \sum_{\mathbf{k}} \exp\{i\mathbf{k}\cdot\mathbf{x}\}\widehat{\psi}(\mathbf{k}). \tag{7}$$

Here $\mathbf{k} = k_0(n,m)$, where $k_0 = 2\pi/L$ is the lowest wavenumber and n and m integers not simultaneously zero. For a given wavenumber ℓ, let $\psi^<$ and $\psi^>$ denote, respectively, the components of ψ spectrally supported by the disk $d = \{\mathbf{k}|k < \ell\}$ and its complement $D = \{\mathbf{k}|k \geq \ell\}$, i.e.

$$\psi^< = \sum_{\mathbf{k}\in d} \exp\{i\mathbf{k}\cdot\mathbf{x}\}\widehat{\psi}(\mathbf{k}), \qquad \psi^> = \sum_{\mathbf{k}\in D} \exp\{i\mathbf{k}\cdot\mathbf{x}\}\widehat{\psi}(\mathbf{k}). \tag{8}$$

For the lower-wavenumber component $\psi^<$, the following inequality holds:

$$\sup_{\mathbf{x}} |\nabla\psi^<| \leq \sum_{\mathbf{k}\in d} k|\widehat{\psi}(\mathbf{k})| \leq \left(\sum_{\mathbf{k}\in d} 1\right)^{1/2} \left(\sum_{\mathbf{k}\in d} k^2|\widehat{\psi}(\mathbf{k})|^2\right)^{1/2} = c\frac{\ell}{k_0}\Psi_2^{<1/2} \tag{9}$$

where c is an absolute constant of order unity and $\Psi_2^< = \langle|\nabla\psi^<|^2\rangle/2$ the large-scale energy density associated with the wavenumbers $k < \ell$. In (9) the Cauchy–Schwarz inequality is used in the second step, and the sum $\sum_{\mathbf{k}\in d} 1 \approx \ell^2/k_0^2$ represents the number of wavevectors in d.

On multiplying (2) by ψ and $(-\Delta)^{1/2}\psi$ and taking the spatial averages of the resulting equations over the domain, noting from the conservation laws that the nonlinear terms identically vanish, one obtains evolution equations for Ψ_1 and Ψ_2,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_1 = -2\mu\Psi_2 + \langle f\psi\rangle, \tag{10}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_2 = -2\mu\Psi_3 + \langle f(-\Delta)^{1/2}\psi\rangle, \tag{11}$$

where $\Psi_3 = \langle|(-\Delta)^{3/4}\psi|^2\rangle/2$. Using the Cauchy–Schwarz and Young inequalities, one obtains the upper bounds on the injection terms in (10) and (11):

$$\begin{aligned}\langle f\psi\rangle &\le \langle|(-\Delta)^{1/2}\psi|^2\rangle^{1/2}\langle|(-\Delta)^{-1/2}f|^2\rangle^{1/2} \le \mu\Psi_2 + \mu^{-1}F_{-2},\\ \langle f(-\Delta)^{1/2}\psi\rangle &\le \langle|(-\Delta)^{3/4}\psi|^2\rangle^{1/2}\langle|(-\Delta)^{-1/4}f|^2\rangle^{1/2} \le \mu\Psi_3 + \mu^{-1}F_{-1},\end{aligned} \tag{12}$$

where the norms of f are defined by $F_{-1} = \langle|(-\Delta)^{-1/4}f|^2\rangle/2$ and $F_{-2} = \langle|(-\Delta)^{-1/2}f|^2\rangle/2$. These norms are assumed to be finite. Substituting (12) in (10) and (11) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_1 \le -\mu\Psi_2 + \mu^{-1}F_{-2}, \tag{13}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_2 \le -\mu\Psi_3 + \mu^{-1}F_{-1}. \tag{14}$$

Let the overline denote the asymptotic average (existence is assumed). One can deduce the following upper bounds for $\overline{\Psi}_2$ and $\overline{\Psi}_3$:

$$\overline{\Psi}_2 \le \mu^{-2}\overline{F}_{-2}, \tag{15}$$
$$\overline{\Psi}_3 \le \mu^{-2}\overline{F}_{-1}. \tag{16}$$

Ineqs. (15) and (16) apply to both the unbounded and bounded domains. In the study of turbulent transfer, the spectral support of f is required to be in the intermediate wavenumber region, so as to render inverse- and direct-transfer ranges free of sources. The limit of infinite domain ($k_0 \to 0$), for which the classical theory is formulated, is taken in a straightforward manner: in the limit $L \to \infty$, the dissipation coefficient μ, the injection densities, and the forcing characteristic length scale (cf. [30]) are held fixed. In this limit both F_{-1} and F_{-2} are bounded if $F = \langle|f|^2\rangle/2$ is bounded since $F_{-2} \le k_{\mathrm{m}}^{-1}F_{-1} \le k_{\mathrm{m}}^{-2}F$, where k_{m} is the minimum wavenumber of the spectral support of f. A persistent inverse cascade of Ψ_1 toward ever-lower wavenumbers necessarily requires $\mathrm{d}\Psi_1/\mathrm{d}t > 0$, which, by (13), implies $\Psi_2 < \mu^{-2}F_{-2}$. Therefore, since the main concern of the subsequent analyses is the realizability of an inverse cascade, for the rest of this paper the (instantaneous) energy Ψ_2 is assumed to be bounded in the limit $k_0 \to 0$.

3 Diminishing inverse transfer

This section reports the main result of this paper. In the subsequent analyses, rigorous estimates are supplemented by the usual assumption of power-law scaling for the energy spectrum. An inverse-transfer range $\Psi_2(k) = ak^{-\alpha}$, for $k < s$, is assumed. Here s may be taken to be the minimum wavenumber of the spectral support of the forcing. An upper bound for the total transfer of Ψ_1 into the low-wavenumber region $[k_0, \ell \ll s]$, i.e. the inverse flux of Ψ_1 across ℓ, is derived in terms of the kinetic energy Ψ_2 and of the large-scale kinetic energy $\Psi_2^<$. It is shown that this flux diminishes in the limit $\ell/s \to 0$. The implication is that no persistent inverse cascade of Ψ_1 is realizable.

The evolution of $\Psi_1^< = \langle|(-\Delta)^{1/4}\psi^<|^2\rangle/2$ is governed by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Psi_1^< &= -\langle\psi^< J(\psi, (-\Delta)^{1/2}\psi)\rangle - 2\mu\Psi_2^< \\
&= -\langle\psi^< J(\psi^>, (-\Delta)^{1/2}\psi)\rangle - 2\mu\Psi_2^< \\
&= \langle(-\Delta)^{1/2}\psi J(\psi^>, \psi^<)\rangle - 2\mu\Psi_2^< \\
&\le \langle|(-\Delta)^{1/2}\psi||\nabla\psi^>||\nabla\psi^<|\rangle - 2\mu\Psi_2^<,
\end{aligned} \tag{17}$$

where (3) has been used in both the second and third steps. The final triple-product term represents an upper bound for the inverse flux of Ψ_1 across the wavenumber ℓ that drives the large-scale dynamics.

In the limit of infinite domain, no finite rigorous estimates of the nonlinear term in (17) are available. Nevertheless, since the norm $|\nabla\psi^<|$ represents a measure of the large-scale fluid velocity (associated with $k < \ell$), it could heuristically be identified with ${\Psi_2^<}^{1/2}$. Hence, a rough estimate of $|\nabla\psi^<|$ would be $|\nabla\psi^<| \approx 2^{-1/2}c'\langle|\nabla\psi^<|^2\rangle^{1/2} = c'{\Psi_2^<}^{1/2}$, where c' is a constant. Substituting this estimate into (17) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_1^< \le 2c'{\Psi_2^<}^{1/2}\Psi_2^{1/2}{\Psi_2^>}^{1/2} - 2\mu\Psi_2^< \le 2c'{\Psi_2^<}^{1/2}\Psi_2 - 2\mu\Psi_2^<, \tag{18}$$

where $\Psi_2^> = \langle|\nabla\psi^>|^2\rangle/2$ is the small-scale energy, associated with $k \ge \ell$. Since Ψ_2 is bounded and since its spectrum $\Psi_2(k)$ is assumed to obey power-law scaling in the inverse-transfer region, $\Psi_2^<$ necessarily diminishes as $\ell \to 0$. Therefore, the quantity $2c'{\Psi_2^<}^{1/2}\Psi_2$, which bounds the inverse flux of Ψ_1 across ℓ, can become arbitrarily small for sufficiently low ℓ. Thus, no persistent inverse cascade of Ψ_1 is realizable provided that the foregoing heuristic estimate of $|\nabla\psi^<|$ can be assumed.

A rigorous version of the above calculation can be deduced for the bounded

case. By applying (9) to (17) one obtains

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Psi_1^< &\le \sup_{\mathbf{x}} |\nabla\psi^<| \langle |(-\Delta)^{1/2}\psi||\nabla\psi^>|\rangle - 2\mu\Psi_2^< \\ &\le 2c\frac{\ell}{k_0}\Psi_2^{<1/2}\Psi_2^{1/2}\Psi_2^{>1/2} - 2\mu\Psi_2^< \le 2c\frac{\ell}{k_0}\left(\frac{\Psi_2^<}{\Psi_2}\right)^{1/2}\Psi_2^{3/2} - 2\mu\Psi_2^<.\end{aligned} \tag{19}$$

The difference between (18) and (19) is the presence of the ratio ℓ/k_0 in the latter. Hence, the arguments in the preceding paragraph go through without change if the ratio ℓ/k_0 is either held fixed or allowed to grow at some certain rate as $\ell \to 0$. This condition can be stated more explicitly by making use of the assumed spectrum $\Psi_2(k) = ak^{-\alpha}$ for $k < s$. In the limit $k_0 \to 0$, the large-scale energy $\Psi_2^<$ and the total energy Ψ_2 can be estimated as

$$\Psi_2^< = a\int_0^\ell k^{-\alpha}\,\mathrm{d}k = \frac{a}{1-\alpha}\ell^{1-\alpha}, \tag{20}$$

$$\Psi_2 \ge a\int_0^s k^{-\alpha}\,\mathrm{d}k = \frac{a}{1-\alpha}s^{1-\alpha}, \tag{21}$$

where $\alpha < 1$, in accord with the boundedness of energy. It follows that

$$2c\frac{\ell}{k_0}\left(\frac{\Psi_2^<}{\Psi_2}\right)^{1/2}\Psi_2^{3/2} \le 2c\frac{\ell}{k_0}\left(\frac{\ell}{s}\right)^{(1-\alpha)/2}\Psi_2^{3/2}. \tag{22}$$

Given a bounded Ψ_2 and a ratio ℓ/k_0 that does not grow as rapidly as $(s/\ell)^{(1-\alpha)/2}$ as ℓ becomes small, this upper bound on the inverse flux of Ψ_1 clearly diminishes as $\ell/s \to 0$. Hence, for a sufficiently wide inverse-transfer range $[\ell, s]$, the advective nonlinearities of SQG turbulence are incapable of transferring a significant amount of the injection of Ψ_1 to the low-wavenumber region $[k_0, \ell]$.

It is interesting to generalize the above calculations to other models of incompressible fluid turbulence in two dimensions. Pierrehumbert et al. [21] propose to consider the so-called α-turbulence models, for which the unforced and inviscid dynamics are governed by

$$\partial_t(-\Delta)^{\alpha/2}\psi + J(\psi, (-\Delta)^{\alpha/2}\psi) = 0, \tag{23}$$

where α is a positive number. The two invariants of this system are $\Psi_\alpha = \langle|(-\Delta)^{\alpha/4}\psi|^2\rangle/2$ and $\Psi_{2\alpha} = \langle|(-\Delta)^{\alpha/2}\psi|^2\rangle/2$. The inverse transfer of Ψ_α across ℓ can be estimated as

$$
\begin{aligned}
|\langle\psi^< J(\psi,(-\Delta)^{\alpha/2}\psi)\rangle| &= |\langle(-\Delta)^{\alpha/2}\psi J(\psi^>,\psi^<)\rangle| \\
&\le \langle|(-\Delta)^{\alpha/2}\psi||\nabla\psi^>||\nabla\psi^<|\rangle \\
&\le \sup_{\mathbf{x}} |\nabla\psi^<|\langle|(-\Delta)^{\alpha/2}\psi||\nabla\psi^>|\rangle \\
&\le 2c\frac{\ell}{k_0}\Psi_2^{<\,1/2}\Psi_{2\alpha}^{1/2}\Psi_2^{>\,1/2}.
\end{aligned} \tag{24}
$$

There are cases for which $\Psi_2^< \to 0$ in the limit $\ell \to 0$, leading to the diminishing of the inverse flux across ℓ. For example, if the direct-cascading candidate $\Psi_{2\alpha}$ remains bounded[2] in the limit $k_0 \to 0$ and if $\alpha < 1$, then $\Psi_2^< \le \ell^{2-2\alpha}\Psi_{2\alpha}^< = \ell^{2-2\alpha}\langle|(-\Delta)^{\alpha/2}\psi^<|^2\rangle/2$, which certainly vanishes as $\ell \to 0$ because $\Psi_{2\alpha}^<$ ($\le \Psi_{2\alpha}$) is bounded. On the other hand, $\Psi_2^>$ converges toward the low wavenumbers for the same reason. There remains the modest assumption that $\Psi_2^>$ also converges toward the high wavenumbers. Given all these, the preceding arguments of no inverse cascade go through without change.

Remark 1. For technical reasons, it is difficult to generalize the analyses in this section to the truly unbounded case (i.e. the case $\ell/k_0 \to \infty$, regardless of the ratio ℓ/s). The main difficulty lies in the intrinsic domain-size dependence of the nonlinear term, thereby making its rigorous estimates, such as the one in (19) and the subsequent estimate (22), diverge in the limit $k_0 \to 0$.

Remark 2. If the viscous operator $\mu(-\Delta)^{1/2}$ is replaced by one of higher degree, as is often done in numerical simulations, then the boundedness of energy in the limit $k_0 \to 0$ cannot be guaranteed. Nevertheless, if boundedness of energy is assumed, then the above calculations go through without change since they do not refer at all to the dissipation mechanism.

Remark 3. The above procedure, when applied to the nonlinear term of the two-dimensional Navier–Stokes equations, yields an estimate of the inverse energy flux that is energy-dependent. This behaviour makes it challenging to estimate the inverse energy transfer since the energy is supposed to grow with progressively wider inverse-transfer range. Nevertheless, it can be shown that for the Kolmogorov–Kraichnan $k^{-5/3}$ energy spectrum the energy that gets transferred onto k_0 can be bounded from above by a constant independent of the width of the inverse-cascading range, i.e. independent of k_0. This result, together with related issues such as the Kolmogorov constant and the Kraichnan conjecture of energy condensation at k_0, is the subject of a separate study.

Remark 4. Recently, Eyink [10] raised the possibility of energy condensation in bounded two-dimensional Navier–Stokes turbulence, a conjecture by Kraichnan [14], and argued that this possibility cannot be ruled out in some

[2] This is certainly the case for SQG ($\alpha = 1$) and Navier–Stokes ($\alpha = 2$) turbulence, for which the energy and enstrophy, respectively, remain bounded in the limit of infinite domain.

recent theoretical results. Kraichnan [14], in an attempt to apply his dual-cascade hypothesis (initially formulated for unbounded fluids) to bounded turbulence, predicts that the inverse energy cascade, upon reaching k_0, deposits energy to this wavenumber, and that this process continues until growth of energy at k_0 is limited by its own dissipation, resulting in what may be termed an "energy condensate". The energy dissipation by this condensate alone is supposed to account for virtually all the energy injection, so that the energy condensate is also an enstrophy condensate, although the latter is of a lesser degree. The realization of such a "singular" energy and enstrophy concentration at k_0 (or around k_0) is required to maintain the proposed dual cascade in the bounded case. Recent numerical results seemed to suggest otherwise: as the turbulence approaches a steady state, a k^{-3} energy spectrum forms at the large scales [3,27]. Nevertheless, the a priori exclusion of the Kraichnan scenario by some recent theoretical studies, such as Constantin et al. [7], Tran and Shepherd [29], and Kuksin [16], does not seem to be fully justified. The SQG dynamics allows for no possibility of such a condensate.

4 Approach to steady dynamics and spectral distribution of energy

This section features some physical interpretations of the results derived in the preceding section. The non-cascading dynamics of SQG turbulence is discussed together with a review of the constraint on the spectral distribution of energy derived by Tran [24].

It is customary in the study of 2D turbulence to consider the scenario in which the fluid is driven around a wavenumber s by steady injections $\langle f\psi\rangle = \epsilon$ and $\langle f(-\Delta)^{1/2}\psi\rangle = s\epsilon$. The result in the preceding section implies that an inverse transfer of a nonzero fraction of ϵ to sufficiently low wavenumbers requires that the energy spectrum in the inverse-transfer region be no shallower than ak^{-1}. But then the energy would grow at least as rapidly as $a\ln(s/\ell)$ as $s/\ell \to \infty$, eventually leading to a balance between the injection ϵ and the dissipation $2\mu\Psi_2$. This result suggests two plausible routes to the long-time high-Reynolds number dynamics (for some suitably defined Reynolds number). First, if a k^{-1} inverse-cascading range is realized, [3] the inverse flux decreases as it proceeds toward lower wavenumbers since the dissipation of Ψ_1, given by $2\mu\Psi_2$, grows logarithmically. Ψ_1 eventually becomes steady, as described above. Second, suppose that an inverse-cascading range with energy spectrum shallower than k^{-1} is realized. The inverse flux decreases for the reason discussed in the preceding section as it proceeds toward sufficiently low wavenumbers. As a result, growth of Ψ_1 (and of Ψ_2) occurs alongside the existing inverse-transfer range. Eventually the dissipation $2\mu\Psi_2$ reaches the injection ϵ and Ψ_1 becomes

[3] Incidentally, dimensional analyses predict a k^{-1} inverse-cascading range.

steady. If power-law scaling is maintained, the slope of the energy spectrum in the inverse-transfer range approaches -1. The low-wavenumber end of this range ℓ can be calculated from $2\mu a \ln(s/\ell) = \epsilon$.

Near steady dynamics can be achieved after the inverse flux of Ψ_1 across ℓ becomes sufficiently less than its dissipation $2\mu\Psi_2$, which should then be comparable to ϵ. Hence, by replacing Ψ_2 in the expression on the right-hand side of (22) by $\epsilon/2\mu$ and requiring that the resulting expression be no larger than ϵ, one obtains a condition for this near steady picture:

$$\left(\frac{\ell}{k_0}\right)^2 \left(\frac{\ell}{s}\right)^{1-\alpha} \le \frac{\mu^3}{\epsilon}, \tag{25}$$

where a constant factor of order unity has been dropped. For fixed (per-unit-area) injection rate ϵ, forcing wavenumber s, and dissipation coefficient μ, no significant fraction of ϵ can be transferred to wavenumbers $k \le \ell$, where ℓ satisfies (25). Ψ_1 necessarily becomes near steady, with the low-wavenumber region $[k_0, \ell]$ at best weakly excited.

For steady dynamics the balances $2\mu\Psi_2 = \epsilon$ and $2\mu\Psi_3 = s\epsilon$ are achieved. It follows that $s\Psi_2 = \Psi_3$, or in terms of the energy spectrum $\Psi_2(k)$,

$$\int_{k_0}^{\infty} (s-k)\Psi_2(k)\, \mathrm{d}k = 0. \tag{26}$$

This equation can be used to estimate the slopes of the energy spectrum if power-law scaling is assumed for both the inverse- and direct-transfer ranges. For this purpose, let us consider the following spectrum

$$\Psi_2(k) = \begin{cases} ak^{-\alpha} & \text{if } \ell < k < s, \\ bk^{-\beta} & \text{if } s < k < k_\nu, \end{cases} \qquad as^{-\alpha} = bs^{-\beta}, \tag{27}$$

where a, b, α, β are constants, and k_ν is the highest wavenumber in the range $k^{-\beta}$, beyond which the spectrum is supposed to be steeper than $k^{-\beta}$. By substituting this spectrum into (26) and making the respective substitutions $\kappa = k/s$ for $k < s$, and $\kappa = s/k$ for $k > s$, one obtains [24]

$$\int_{\ell/s}^{1} (1-\kappa)\kappa^{-\alpha}\, \mathrm{d}\kappa \approx \int_{s/k_\nu}^{1} (1-\kappa)\kappa^{\beta-3}\, \mathrm{d}\kappa, \tag{28}$$

where the contribution from both $k \le \ell$ and $k \ge k_\nu$ has been dropped. It follows that if $\ell/s \ge s/k_\nu$, then $-\alpha \le \beta - 3$. Hence, the constraint

$$\alpha + \beta \geq 3 \tag{29}$$

holds. Since $\alpha \leq 1$, β satisfies $\beta \geq 2$, meaning that for $k > s$, the spectrum $\Psi_3(k)$ of Ψ_3 is no shallower than k^{-1}. Hence, the energy dissipation cannot occur mainly at $k \gg s$. Thus there is no direct cascade, a dynamical behaviour consistent with no inverse cascade of Ψ_1.

Remark 5. Although no persistent inverse flux is possible, in the limit $k_0 \to 0$, Ψ_1 could become unbounded, growing at a rate that fluctuates about zero and has a vanishingly small positive average. The unboundedness of Ψ_1 requires that the energy spectrum of the inverse-transfer range have a non-positive slope, so that the spectrum $\Psi_1(k)$ is at least as steep as k^{-1}. For a given set of physical parameters, the issue of whether or not Ψ_1 becomes divergent (in the limits $k_0 \to 0$ and $t \to \infty$) is interesting but is beyond the scope of the present work.

Remark 6. In some sense SQG turbulence is relatively "simple". Given spectrally localized steady injections about the forcing wavenumber s, the dynamics should eventually become non-cascading. For a k^{-1} transient inverse-cascading range, the approach to steady dynamics is rather slow for two reasons. First, because $\Psi_1(k) \propto k^{-2}$, the low-wavenumber end of the inverse-cascading range ℓ proceeds relatively slowly toward lower wavenumbers, even during the early stages for which the inverse cascade is relatively strong ($\mathrm{d}\Psi_1/dt \approx \epsilon$). Second, growth of energy is only logarithmic in ℓ^{-1}, giving rise to rather slow growth of the dissipation of Ψ_1 toward the lower wavenumbers.

Remark 7. Dimensional analyses, without references to any particular dissipation mechanisms, predict a k^{-1} inverse-transfer range and a $k^{-5/3}$ direct-transfer range. The former is consistent with an inverse cascade of Ψ_1, but the latter allows for virtually no energy to get transferred to the small scales.[4] Before a viscous dissipation mechanism is taken into consideration, a $k^{-5/3}$ spectrum means that virtually no energy gets transferred to the high wavenumbers. In the presence of a viscous dissipation operator of the form $\propto (-\Delta)^{\delta}$, for $0 \leq \delta \leq 1/3$, instead of the natural dissipation operator $\mu(-\Delta)^{1/2}$, a $k^{-5/3}$ direct-transfer range means that the spectral energy dissipation scales as $k^{-5/3+2\delta}$, which is no shallower than k^{-1}, thereby allowing for virtually no energy to be dissipated at its high-wavenumber end.

Remark 8. The simultaneous conservation of Ψ_1 and Ψ_2 leads to an increase

[4] These predictions seem to be mutually inconsistent since the inverse-transfer of Ψ_1 via a k^{-1} spectrum is incompatible with the "frozen-in" of energy due to the $k^{-5/3}$ direct-transfer range. This picture is quite contrary to that of two-dimensional Navier–Stokes turbulence, for which a k^{-3} enstrophy-transfer range means that virtually all enstrophy gets transferred away from the forcing region, even before dissipative effects are considered.

in Ψ_4 (enstrophy) when an initial spectral peak spreads out in wavenumber space [24]. This explains the observed formation of strong "fronts" in numerical simulations of SQG turbulence [8,9,18], even in the absence of a direct energy cascade as discussed above.

5 Numerical results

This section reports results from numerical simulations that illustrate the diminishing inverse transfer and no cascades of SQG dynamics. Numerical studies in the literature have thus far failed to recognize these properties of SQG turbulence (see for example [19,21,23,26,28]). Tran and Bowman [28], however, notice that Ψ_1 is "reluctant" to cascade to the large scales, even when the natural dissipation operator $\mu(-\Delta)^{1/2}$ is replaced by ones with higher degrees, allowing for relatively weaker dissipation at the large scales.

Equation (2) is simulated in a doubly periodic square of side 2π, where the modal forcing $\widehat{f}(\mathbf{k})$ is nonzero only for those wavevectors $\mathbf{k}$ having magnitudes lying in the interval $K = [9.5, 10.5]$:

$$\widehat{f}(\mathbf{k}) = \frac{s\epsilon}{N} \frac{\widehat{\psi}(\mathbf{k})}{k \sum_{|\mathbf{p}|=k} |\widehat{\psi}(\mathbf{p})|^2}, \tag{30}$$

where $s\epsilon = 1$ is the constant energy injection rate and N the number of discrete wavenumbers in K. The wavenumber $s \approx 10$ is defined such that s^{-1} is the mean of k^{-1} over K. The (constant) injection rate of Ψ_1 is $\epsilon \approx 0.1$. Dealiased 683^2 and 1365^2 pseudospectral simulations (1024^2 and 2048^2 total modes) were performed. For Navier–Stokes turbulence, these resolutions are sufficient to simulate an inverse cascade that carries about a quarter of the energy injection to the large scales via a discernible $k^{-5/3}$ range [24]. For SQG turbulence, even the higher resolution turns out to be insufficient to simulate a noticeable transient inverse cascade. Two dissipation coefficients were used: $\mu = 0.05$ and $\mu = 0.025$. The lower and higher resolutions were used for the stronger and weaker dissipation, respectively. Both simulations were initialized with the spectrum $\Psi_2(k) = 10^{-2}\pi k/(100 + k^2)$.

Figure 1 shows the time-averaged (between $t = 19.7$ and $t = 20.3$)[5] kinetic energy spectrum for the case $\mu = 0.05$. The average energy is $\Psi_2 = 0.99$. The dissipation of Ψ_1, averaged for the same period, is $2\mu\Psi_2 = 0.099$, which

[5] The dissipation time at the forcing wavenumber $s \approx 10$ is $(2\mu s)^{-1} \approx 1$, so that the time $t \approx 20$ is sufficiently long for the evolution of Ψ_2. In fact, it was observed that both Ψ_1 and Ψ_2 became steady at $t \approx 12$.

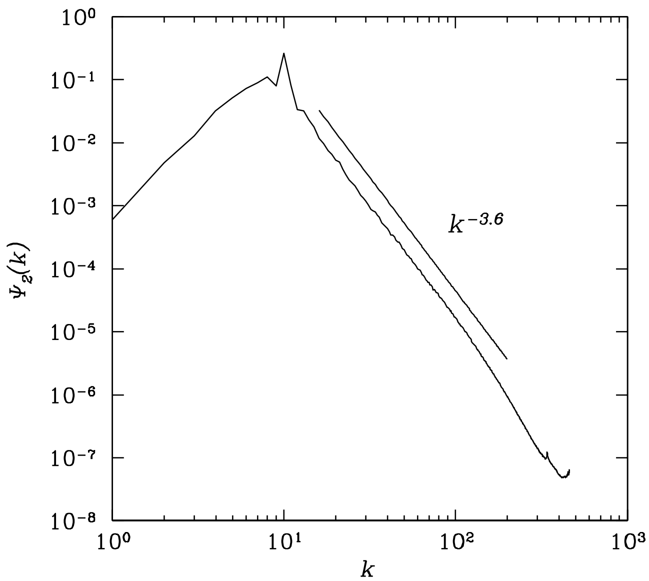

Fig. 1. The time-averaged steady-state energy spectrum $\Psi_2(k)$ vs. k for the dissipation coefficient $\mu = 0.05$. The average energy is $\Psi_2 = 0.99$, implying that the dissipation of Ψ_1, averaged in the same period, is 0.099. This amounts to virtually all of the injection rate $\epsilon \approx 0.1$. Hence, no inverse cascade of Ψ_1 exists and both Ψ_1 and Ψ_2 are steady.

amounts to virtually all the injection rate ≈ 0.1. The energy was observed to increase monotonically from $t = 0$ to $t \approx 12$, by which time $\mathrm{d}\Psi_1/\mathrm{d}t \approx 0$; no significant inverse transfer was observed throughout the period. Instead, growth of energy occurs mainly around the forcing region. The steady spectrum is relatively shallow in the low-wavenumber range (see Fig. 1), meaning that the lowest wavenumbers are virtually unexcited. The small-scale energy spectrum scales as $k^{-3.6}$, so that the spectrum $\Psi_3(k)$ of the energy dissipation agent Ψ_3 scales as $k^{-2.6}$. This scaling means that the energy dissipation occurs mainly around the forcing region, consistent with the weak inverse transfer.

A somewhat stronger transient inverse transfer was observed for the case $\mu = 0.025$. Fig. 2 shows a near steady kinetic energy spectrum averaged between $t = 15.1$ and $t = 15.6$. The average energy is $\Psi_2 = 1.96$. The dissipation of Ψ_1 averaged for the same period is $2\mu\Psi_2 = 0.098$, which amounts to virtually all of the injection rate ≈ 0.1. The large-scale spectrum is better "filled up" than that of the previous case, due to a stronger inverse transfer during the transient phase. Nevertheless, no significant fraction of ϵ reaches the lowest wavenumbers. The energy dissipation occurs around the forcing region, as is evident from the steep small-scale spectrum (see Fig. 2).

It is expected that for higher resolutions (so that simulations with smaller μ are possible), transient inverse fluxes can become more noticeable and steeper inverse-transfer ranges can be realized. Given limited resolutions, Tran and Bowman [28] use high-degree dissipation operators $\propto \Delta$ and $\propto (-\Delta)^{3/2}$. In

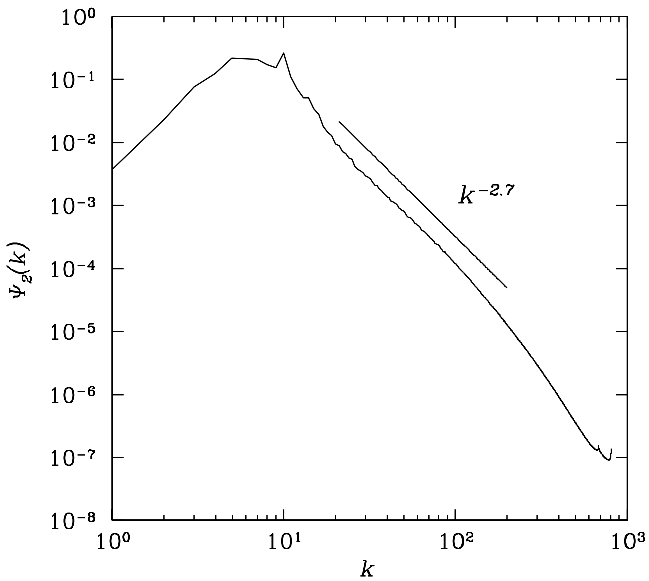

Fig. 2. The time-averaged near steady-state energy spectrum $\Psi_2(k)$ vs. k for the dissipation coefficient $\mu = 0.025$. The average energy is $\Psi_2 = 1.96$, implying that the dissipation of Ψ_1, averaged in the same period, is 0.098. This amounts to most of the injection rate $\epsilon \approx 0.1$.

both cases, weak inverse cascades are observed, but the transient energy spectra of the inverse-transfer region are considerably shallower than k^{-1}. These spectra cannot support a persistent inverse cascade, as argued in Section 3. However, an interesting possibility arises. Due to both the inability of the nonlinear transfer to sustain a constant (wavenumber-independent) inverse flux and $\mathrm{d}\Psi_1/\mathrm{d}t > 0$, growth of Ψ_1 ought to occur within the existing inverse-transfer range, thereby causing this range to become steeper. This process may continue until the inverse-transfer range eventually exceeds the k^{-1} threshold. Because of the high degrees of viscosity, such a slope steepening causes insignificant increase in the dissipation of Ψ_1. As a result, a positive growth rate $\mathrm{d}\Psi_1/\mathrm{d}t$ could be maintained for a steeper-than-k^{-1} inverse-transfer range, which might then be able to support a persistent inverse cascade.

6 Conclusion

In this paper, the advective transfer of SQG turbulence is studied. The main result obtained is an upper bound for the inverse flux of the inverse-cascading candidate Ψ_1. This upper bound diminishes as the flux proceeds toward sufficiently low wavenumbers, thereby ruling out the existence of a persistent inverse cascade. The unrealizability of an inverse cascade entails that there is no direct cascade. This is the first rigorous example of the dynamics of incompressible fluids in two dimensions that exhibits no cascades.

There are two essential features of SQG turbulence that facilitate the proof of non-cascading dynamics. First, the inverse flux of Ψ_1 across a low wavenumber ℓ can be uniformly (in time) estimated in terms of the kinetic energy Ψ_2 and of $\Psi_2^<$, the large-scale energy associated with the low-wavenumber region $k < \ell$. Second, the energy Ψ_2 remains bounded in the limit of infinite domain. The former is an intrinsic property of the advective nonlinear term, and the latter is due to the hypoviscous nature of the dissipation of SQG dynamics. The boundedness of energy means that if the turbulence is driven at some fixed energy density rate around some fixed wavenumber s, virtually no energy can be acquired by wavenumbers $k \ll s$. This means that in the limit $\ell/s \to 0$, $\Psi_2^<$ can become arbitrarily small, leading to an arbitrarily small inverse flux of Ψ_1 across ℓ. Hence, for sufficiently low wavenumber ℓ no significant inverse transfer of Ψ_1 across ℓ is possible.

Numerical simulations of SQG turbulence were performed. The results show no significant inverse transfer of Ψ_1 to the large scales, thereby lending strong support to the prediction of no cascades. The large-scale energy spectra are shallower than k^{-1}, and the small-scale spectra are steeper than k^{-2}, consistent with the theoretical prediction.

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